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IDENTIFICATION THE HEAT SOURCE DEPENDENT ON A SPATIAL VARIABLE, REGULARIZATION AND CONTROL OF NOISE LEVEL IN CASE PERTURBATION BY SPECTRAL TRUNCATION AND KRYLOV METHODS

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Abstract

This paper deals with regularization and noise level control of source depending only on the spatial variable in the heat equation. In this problem, the Krylov method is employed to get the regularization solution and control noise level in case perturbation. Various numerical tests are given to verify the efficiency of the proposed method and until what point the method resist at the perturbations.

Keywords: Ill-posed problems; inverse problems; regularization; heat Source; Krylov subspace method; Krylov projection method.

I. INTRODUCTION

The inverses problems are a very active area of mathematical and numerical research over the past 50 years, with applications of significant economic and societal impact.

We seek to regularized the solution of an inverse problem identification of source and how to control noise level for noisy data. These problems are ill-posed, and for a stable numerical approximation of the solution some regularization techniques have to be applied. The formal solution is written as a Fourier series with high frequency (via its spectral eigen function expansion). We suggest a regularization procedure based on the Krylov method. We give a theoretical analysis of these methods, and some numerical examples to show the accuracy. With a live control in Krylov subspace, the exact controllability for the heat equation is impractical and we will be content with an approximate control result. Our objective will then be to determine the cost of the approximate control.

Position of the problem

We consider the following inverse problem: Find the pair of functions (u(x,t), f(x)) which satisfies: (cf.[1])

Equation (1) showing partial differential equations and boundary conditions for u(x,t) and f(x).

u(x,t) is the body temperature at a given point x of the axis at a given time t, and f(x) is the unknown source of heat depending only on the spatial variable x.

This problem is called the inverse problem of identification of unknown source.

The boundary conditions:

Equation (2) showing boundary conditions for u(x,t) at x=0 and x=1.

The final condition: u(x,1) = g(x), where g is a given measurement input internal. In applications, the input data g(x) can only be measured, and there will be measured data function g\_delta(x) which is merely in L^2(0,1) and satisfies

Equation (3) showing the norm of the difference between g and g\_delta.

where the constant  $\delta > 0$  represents a noise level of input data.

By the separation of variables, the solution of Problem (1) can be obtained as follows:

$$u(x, t) = \sum_{n=1}^{\infty} \frac{1 - e^{-n^2 \pi^2 t}}{n^2 \pi^2} f_n e_n \quad (4)$$

where

$$\{e_n = \sqrt{2} \cos n\pi x, (n = 1, 2, \dots)\} \quad (5)$$

is an orthogonal basis in  $L^2(0, 1)$ , and

$$f_n = \sqrt{2} \int_0^1 f(x) \cos(n\pi x) dx. \quad (6)$$

Making use of the final condition :

$$g(x) = \sum_{n=1}^{\infty} (g, e_n) e_n = \sum_{n=1}^{\infty} g_n e_n = \sum_{n=1}^{\infty} \frac{1 - e^{-n^2 \pi^2}}{n^2 \pi^2} f_n e_n \quad (7)$$

and defining the operator  $K : f \rightarrow g$ , we obtain:

$$g(x) = Kf(x) = \sum_{n=1}^{\infty} \frac{1 - e^{-n^2 \pi^2}}{n^2 \pi^2} f_n e_n. \quad (8)$$

It is easy to see that  $K$  is a linear compact operator, and the singular values  $\{\gamma_n\}_{n=1}^{\infty}$  of  $K$  are

$$\gamma_n = \frac{1 - e^{-n^2 \pi^2}}{n^2 \pi^2}, (n = 1, 2, \dots). \quad (9)$$

On the other hand

$$g_n = (g, e_n) = \gamma_n f_n (e_n, e_n) \quad (10)$$

i.e.,

$$f_n = \gamma_n^{-1} g_n. \quad (11)$$

Therefore

$$f(x) = K^{-1}g(x) = \sum_{n=1}^{\infty} \frac{1}{\gamma_n} g_n e_n = \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{1 - e^{-n^2 \pi^2}} g_n e_n. \quad (12)$$

Note that  $\frac{1}{\gamma_n} \rightarrow \infty$  if  $n \rightarrow \infty$ , which makes a small perturbation  $g$  cause the explosion of the solution. So, the problem is ill-posed because the solution does not continuously depend on the initial data. As there is no source of heat which is supplied indefinitely, we posed the question of the applicability of an effective method of truncation for the identification and regularization of the solution. This instability results from the behavior of high frequencies.

We propose In this paper:

- In section 2, We propose a method of spectral truncation to construct a stable approximation of the solution by projecting it onto a space of small size called Krylov space to be able to compute it numerically.
- Section 3 suggest a regularization procedure based on the spectral truncation method and the Krylov method.
- Section 4 give a theoretical analysis of these methods, and some numerical examples to show the accuracy.
- And finally, in Section 4 we some remarks and conclusion.

## II. STABILIZATION AND APPROXIMATION

We now want to address the issue of stabilization in the case of noise-impaired data, because the data available are experimental, which implies the existence of measurement errors. This cause of uncertainty induces a fuzzy image because of the sensitivity of the inverse problems to the uncertainties, and the interpretation of the answers becomes almost impossible and generates a great risk.

We suppose now that the data  $u(\tau) = g$  is tainted by (inaccurate) noise, ie, we have an approximation  $g_\delta$  de  $g : \|g_\delta - g\| \leq \delta$ ,  $\delta$  is the noise level.

## 1. Approaching Problem (1) by the Krylov Method

Let  $H = L^2(0,1)$ . We consider Problem (1): Find the pair of functions  $(u(x,t), f(x))$  that satisfies

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x) & 0 < x < 1, \quad 0 < t \leq 1 \\ u(x,0) = 0, & 0 \leq x \leq 1 \\ \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0, & 0 \leq t \leq 1 \\ u(x,1) = g(x), & 0 \leq x \leq 1. \end{cases}$$

It is easy to see that the pair of functions

$$u(x,t), f(x) = \left( \frac{1 - e^{-\pi^2 t}}{\pi^2} \cos(\pi x), \cos(\pi x) \right) \quad (13)$$

is the exact solution of Problem (1.1). Consequently, the data function is  $g(x) = \frac{1 - e^{-\pi^2}}{\pi^2} \cos(\pi x)$ . You can put the couple in the form of solution:

$$(u(x,t), f(x)) = \left( (1 - e^{-\pi^2 t}) \cos(\pi x), \pi^2 \cos(\pi x) \right) \quad (14)$$

and

$$g(x) = (1 - e^{-\pi^2}) \cos(\pi x). \quad (15)$$

### Approximation (cf.[7]) and (cf.[8])

Either the system  $Au = v \Leftrightarrow u = \varphi(A)v$ . Our goal is to obtain a solution approached this system that is sufficiently precise for the needs and lowest possible cost of calculation.

**The heat source identified is given by equation (12) :**

$$f(x) = K^{-1}g(x) = \sum_{n=1}^{+\infty} \frac{1}{\gamma_n} g_n e_n = \sum_{n=1}^{+\infty} \frac{n^2 \pi^2}{1 - e^{-n^2 \pi^2}} g_n e_n.$$

Which can be written numerically  $f(x) = \sum_{n=1}^N \varphi(\lambda_n) g_n e_n$ , where  $\varphi(s) = \frac{s}{1 - e^{-s}}$ ,  $\lambda_n$ , and  $e_n$  are, respectively,

the eigenvalues and eigenvectors of the matrix of the discretized operator  $A_h$  or  $g(x) = \sum_{n=1}^N g_n e_n$ . So,  $f$  is of the form

$$f = \varphi(A)g = (I_n - \exp(-A))^{-1} Ag \quad (16)$$

where  $f, g \in \mathbb{R}^n$ ,  $A \in M_n(\mathbb{R})$ .

Let  $A$  be the unbounded operator defined by:

$$\begin{cases} \mathfrak{D}(A) = \{u \in H^1(0,1); u'(1) = u'(0) = 0\} \\ Au = -\frac{d^2 u}{dx^2} \end{cases} \quad (17)$$

where  $\mathfrak{D}(A)$  is the domain of definition of the operator  $A$ .

**Proposition 0.1.** *The operator  $A$  is self-adjoint and positive. (cf.[6])*

### Discretization and projection of the solution

After the semi-discretization of the operator  $A$ , we have:

$$A_h = \frac{1}{h^2} \begin{pmatrix} 1 & -1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdot & \cdot & 0 \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & -1 & 2 & -1 \\ 0 & 0 & \cdot & \cdot & 0 & 0 & -1 & 1 \end{pmatrix} \quad (18)$$

The matrix  $A_h$  is tridiagonal and symmetric by construction.

**Proposition 0.2.** *The matrix  $A_h$  is symmetric and positive (cf.[3]).*

## 2. Arnoldi approximation order m of $f(A_h)v$

There exist very effective methods that achieve good approximates  $f_m$  even for fairly small m. Such a method is the Arnoldi method:

- Generate an orthonormal basis  $V_m = [v_1, v_2, \dots, v_m]$  of  $K_m(A_h, v)$  using a Gram-Schmidt procedure that satisfies (cf.[4])

$$V_m^t A_h V_m = H_m, \quad (19)$$

where  $H_m \in M_m(\mathbb{R})$  is an upper Hessenberg matrix.

- The Arnoldi approximation of order m (cf.[2]) and (cf.[5]) is defined as

$$f_m = \|v\| V_m f(H_m) e_1. \quad (20)$$

where  $e_m = [0, 0, \dots, 0, 1]^t$

If  $A$  is Hermitian then  $H_m$  is tridiagonal. Instead of orthonormalizing the vector  $v_m$  against all preceding  $v_1, v_2, \dots, v_{m-1}$ , there exists a three-term recurrence involving only  $v_{m-2}$ ,  $v_{m-1}$  and  $v_m$ . This method is called the Lanczos method and in comparison to the Arnoldi method.

- Computation cost decreases rapidly (since only 2 orthogonalizations per time step are necessary).

## III. REGULARIZATION BY SPECTRAL TRUNCATION AND KRYLOV METHOD

To construct a stable approximation for (12), one truncates the series of fourier and one takes only the finite part.

**Definition 0.3.** For  $N > 0$ , we define the regularized solution of the problem (12) for exact (respectively inaccurate) data as follows:

$$f_N = \sum_{k=1}^N \left( \frac{\lambda_k}{1 - e^{-\tau \lambda_k}} \right) \langle g, \xi_k \rangle \xi_k, \quad (21)$$

$$f_N^\delta = \sum_{k=1}^N \left( \frac{\lambda_k}{1 - e^{-\tau \lambda_k}} \right) \langle g_\delta, \xi_k \rangle \xi_k, \quad (22)$$

This method is known as *spectral truncation* which eliminates the high frequencies responsible for instability.

**Theorem 0.4.** We suppose  $f \in B(p, E) = \{f \in D(A^p) : \|A^p f\| \leq E\}$ ,  $p > 0$ , and let  $\lambda_{N+1} \approx \left(\frac{E}{\delta}\right)^{1/(2+p)}$ , then we have the following error estimate:

$$\|f - f_N^\delta\| \leq K \delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}}, \quad (23)$$

where  $K = (1 + M) = 1 + \frac{1}{1 - e^{-\tau \lambda_1}}$ .

*Proof.* Note

$$\begin{aligned} \omega_k &= \frac{\lambda_k}{1 - e^{-\tau \lambda_k}} \leq \frac{\lambda_k}{1 - e^{-\tau \lambda_1}} = M \lambda_k, \\ g_k &= \langle g, \xi_k \rangle, \quad g_k^\delta = \langle g_\delta, \xi_k \rangle. \end{aligned}$$

Using the triangular inequality, we can write

$$\|f - f_N^\delta\| = \|f - f_N + f_N - f_N^\delta\| \leq \|f - f_N\| + \|f_N - f_N^\delta\| = \Delta_1 + \Delta_2. \quad (24)$$

$$\Delta_1^2 = \|f - f_N\|^2 = \left\| \sum_{k=1}^{\infty} f_k \xi_k - \sum_{k=1}^N f_k \xi_k \right\|^2 = \sum_{k=N+1}^{\infty} |f_k|^2. \quad (25)$$

$$\Delta_2^2 = \|f_N - f_N^\delta\|^2 = \left\| \sum_{k=1}^N \omega_k g_k \xi_k - \sum_{k=1}^N \omega_k g_k^\delta \xi_k \right\|^2 = \sum_{k=1}^{N+1} \omega_k^2 |g_k - g_k^\delta|^2. \quad (26)$$

$$\Delta_1^2 = \sum_{k=N+1}^{\infty} \lambda_k^{-2p} \lambda_k^{2p} |f_k|^2 \leq \lambda_{N+1}^{-2p} \sum_{k=N+1}^{\infty} \lambda_k^{2p} |f_k|^2 \leq \lambda_{N+1}^{-2p} E^2. \quad (27)$$

$$\Delta_2^2 = \sum_{k=1}^{N+1} \omega_k^2 |g_k - g_k^\delta|^2 \leq \lambda_{N+1}^2 M^2 \sum_{k=1}^{N+1} |g_k - g_k^\delta|^2 \leq \lambda_{N+1}^2 M^2 \delta^2. \quad (28)$$

which implies

$$\Delta_1 + \Delta_2 \leq \lambda_{N+1}^{-2p} E + \lambda_{N+1}^2 M \delta \approx \left( \left( \frac{E}{\delta} \right)^{\frac{1}{2+p}} \right)^{-p} E + M \delta \left( \frac{E}{\delta} \right)^{\frac{2}{2+p}} = (1 + M) E^{\frac{2}{2+p}} \delta^{\frac{2p}{2+p}}. \quad (29)$$

□

Identity (22) indicates that  $f_N^\delta(x)$  is close to the exact solution  $f(x)$  when the parameter  $N$  becomes very large. On the other hand, if the parameter  $N$  is fixed,  $f_N^\delta(x)$  is bounded. So the positive integer  $N$  plays the role of a regularization parameter.

*Remark 0.5.* If  $p > 0$ ,  $\|f - f_N^\delta\| \leq (1 + M) E^{\frac{2}{2+p}} \delta^{\frac{2p}{2+p}} \rightarrow 0$  as  $\delta \rightarrow 0$ . Hence  $f_N^\delta(x)$  can be viewed as the approximation of the exact solution  $f(x)$ .

#### IV. NUMERICAL RESULTS

We give the numerical results by the Krylov method for the exact data and the noisy data to see the efficiency of the method in the face of a perturbation. the following perturbation:

$$g_\delta = g + \varepsilon \times \text{randn}(\text{size}(g)) \quad (30)$$

with  $\|g - g_\delta\|_{L^2(0,1)} \leq \delta$  where the constant  $\delta > 0$  represents the noise level of the input data.

The function "*Randn*( $\cdot$ )" generates arrays of random numbers whose elements are normally distributed with mean 0, variance  $\sigma^2 = 1$ , and standard deviation  $\sigma = 1$ . "*Randn*(*size*( $g$ ))" returns an array of random entries that is the same size as  $g$ . We take in practical

$$\delta = \|g - g_\delta\|_2 \quad (31)$$

and use Trapezoid's rule to approach the integral and choose the sum of the front  $M$  terms to approximate the solution . After considering an equidistant grid

$$0 = x_0 < x_1 < \dots < x_M = 1, \quad x_k = \frac{k}{M}, \quad k = 0, \dots, M$$

with

$$h = \frac{1}{M} \text{ and } M \text{ is the constant parameter.}$$

Identity (22) indicates

$$f_N^\delta(x) = \int_0^1 2 \sum_{n=1}^N \frac{1}{\sigma_n} g_\delta(s) \cos(n\pi s) \cos(n\pi x) ds \quad (32)$$

At the point  $x_i$  we have

$$f_N^\delta(x_i) = 2 \sum_{k=1}^M \sum_{n=1}^N \frac{1}{\sigma_n} g_\delta(x_k) \cos(n\pi x_k) \cos(n\pi x_i) h \quad (33)$$

Knowing that

$$\|f\|_2 = \sqrt{\int_0^1 f(x)^2 dx}$$

and

$$\int_0^1 f(x)^2 dx = \frac{1}{N} \sum_{i=1}^N f_i^2.$$

So replacing  $f$  by  $g - g_\delta$  and by calculating numerically with the trapezoid method, we obtain:

$$\delta = \|g - g_\delta\|_2 = \left( \frac{1}{N} \sum_{i=1}^N (g_i - g_{\delta,i})^2 \right)^{\frac{1}{2}} \quad (34)$$

To test the accuracy of the approximate solutions, we use error defined as:

$$\theta_m = \|f - f_m^\delta\| \leq 2\|g\| \sum_{k=m}^{+\infty} |h_k|$$

$$\theta_m \leq 2\|g\| \left( 2 \max \left( M_1, 4 + \frac{a(\rho_1 + c)}{\mu} \right) \times \frac{\left( (c + \alpha)^2 \left( 1 + \frac{4\pi^2}{a^2 \alpha^2} \right) \right)^{\frac{-m}{2}}}{1 - \left( (c + \alpha)^2 \left( 1 + \frac{4\pi^2}{a^2 \alpha^2} \right) \right)^{\frac{-1}{2}}} + \right.$$

$$\left. \frac{8\pi}{a} \left| \frac{\phi\left(c + \frac{2\pi i}{a}\right)^{-m-1}}{\left(1 + \phi\left(c + \frac{2\pi i}{a}\right) - 1\right)\left(1 - \phi\left(c + \frac{2\pi i}{a}\right) - 1\right)^2} \right| + \right.$$

$$\left. \frac{8\pi}{a} \left| \frac{\phi\left(c - \frac{2\pi i}{a}\right)^{-m-1}}{\left(1 + \phi\left(c - \frac{2\pi i}{a}\right) - 1\right)\left(1 - \phi\left(c - \frac{2\pi i}{a}\right) - 1\right)^2} \right| \right)$$

The error was published in a [hypertext link](#) in the last part of the Bibliography.

### 1. exact data in case of Neumann condition

We took  $N = 200$ ,  $m = 100$ , and as the starting vector of the Lanczos Algorithm (cf.[5])  $v = g$ , with  $g \in \mathbb{R}^N$ , which is the test function.

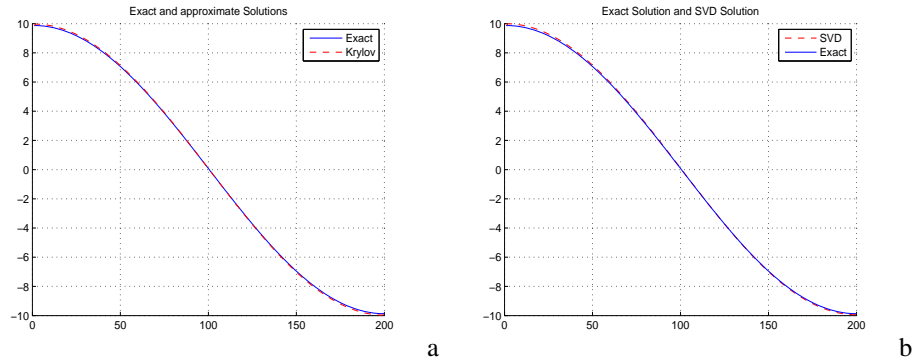


Figure 1: Graphical representation:(a) exact solution and approximation of Krylov, (b) exact solution and singular value decomposition (SVD) solution.

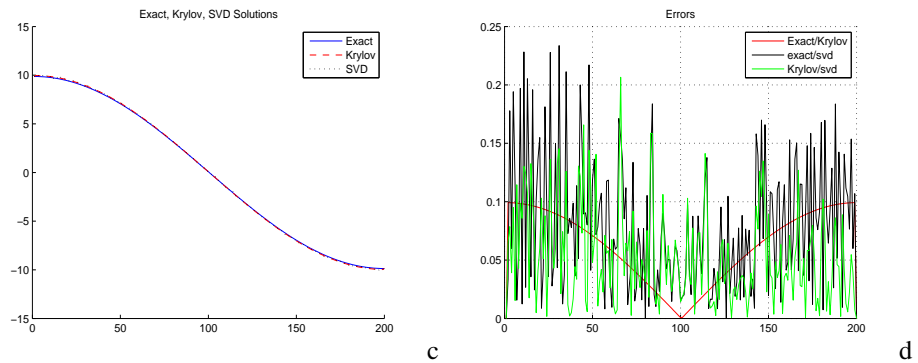


Figure 2: Graphical representation:(c) exact solution, approximation of Krylov and SVD solution, (d) respective errors.

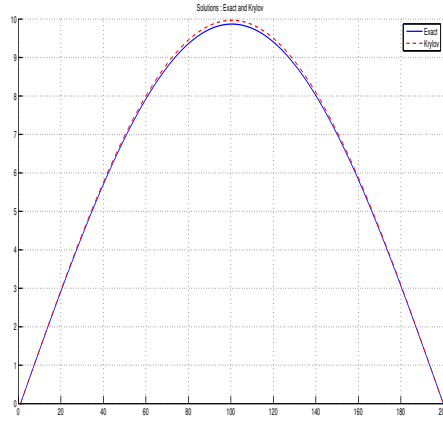
*Remark 0.6.* • The SVD (Singular Value Decomposition) is the method used by Matlab to calculate the matrix functions.

- The advantage of using this method of Lanczos is due to its ability to allow the calculation of a small part of the spectrum without having to calculate the entire spectrum.

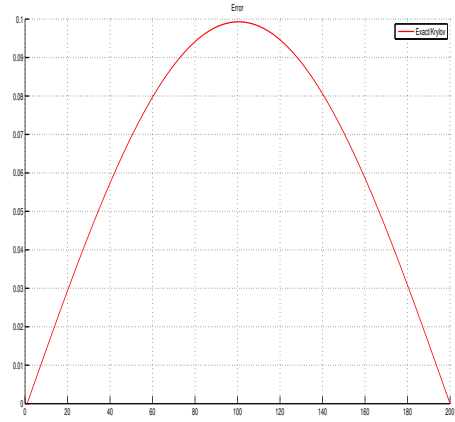
## 2. exact data in case of Dirichlet condition

For case of Dirichlet condition, the matrix  $A_h = \frac{1}{h^2} \text{Tridiag}[-1, 2, -1] \in M_N(\mathbb{R})$  is tridiagonal, symmetric and positive definite matrix.

We took  $N = 200$ ,  $m = 100$ , and as the starting vector of the Lanczos Algorithm  $v = g$ , with  $g \in \mathbb{R}^N$ , which is the test function.



e

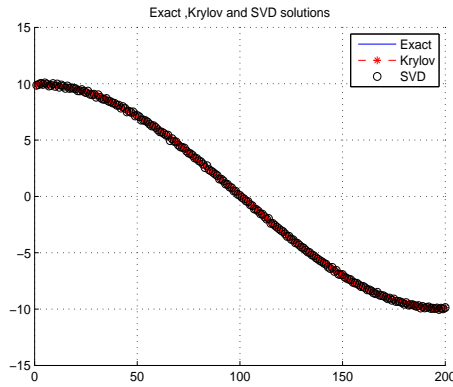


f

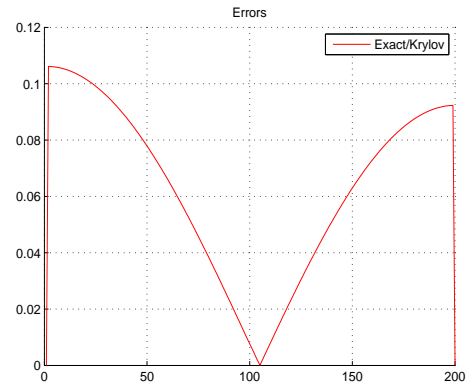
Figure 3: Graphical representation:(e) exact solution and approximation of Krylov, (f) error between the exact solution and approximation of Krylov.

## 3. noisy data in case of Neumann condition

We took  $N = 200$ ,  $m = 100$ , and as starting vector of Lanczos Algorithm  $v = g$ , with  $g \in \mathbb{R}^N$  (which is the test function), and  $\varepsilon \leq 10^{-3}$  we have:



a



b

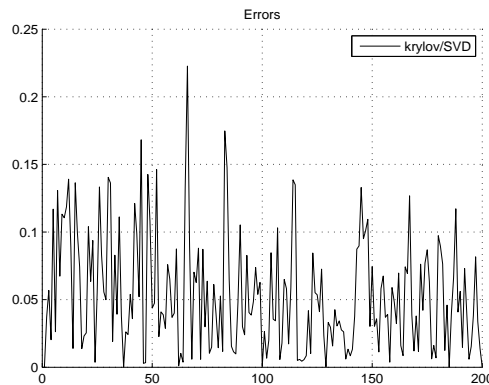


Figure 4: Graphical representation: (a) exact solution, approximation of Krylov and SVD solution (b) error between exact solution and approximation of Krylov (c) errors between approximation of Krylov and SVD solution

Errors can be seen that do not amplify during the course of the algorithm, and the method remains stable. Perturbation techniques show that the solutions of the perturbed system are close to the solution of the unperturbed system with the precision control parameter to  $10^{-3}$  close.

*Remark 0.7.* The problem (1) from the numerical point of view is more difficult when comparing with the Dirichlet problem. Indeed, the operator  $A$  is self-adjoint, positive and  $\lambda = 0$  is an eigenvalue. This induces a technical difficulty when we pass to the discrete problem.

## V. DISCUSSION

In this study, a convergent and stable reconstruction of an unknown source heat has been obtained using two regularizing methods: spectral truncation method and Krylov method. We have shown that the Krylov method which was known for its theoretically regulating effect is even better numerically. We limited ourselves to a few cases of perturbation and controlled the noise level by the regularization of the solution. This problem has a real impact on the reconstruction of fuzzy images of medical X-ray.

Future work will involve the problem of approximating a solution of an ill-posed biparabolic problem in the abstract setting and estimate error between the exact solution and approximation solution.

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